Painlevé, Klein & the icosahedron

P. Boalch (ENS, Paris)

Madrid September 2006

Main theme

"Icosahedral" solutions of linear & nonlinear (ordinary) differential equations

- although we will try to go beyond the icosahedron

Classical example

Icosahedral rotation group A_5 of order 60

$$A_5 = \Delta_{235} = \langle a, b, c \mid a^2 = b^3 = c^5 = abc = 1 \rangle$$

- natural to look for ODEs on $\mathbb{P}^1 \setminus 3$ points with monodromy A_5

$$A_5 \subset SO_3(\mathbb{R}) \subset SO_3(\mathbb{C}) \cong PSL_2(\mathbb{C})$$

- look for connections on rank 2 vector bundles with projective monodromy A_5

Schwarz's List (1873)

Gauss hypergeometric equation

$\Rightarrow$ logarithmic connection $\left(\frac{A_1}{z} + \frac{A_2}{z-1}\right)dz$

- A_1, A_2 2×2 rank 1 matrices

  $\{$ (twist by log. connection on line bundle)

- $A_1, A_2 \in sl_2(\mathbb{C})$ (2×2 trace free)

[connection on trivial principal $SL_2(\mathbb{C})$ bundle over $\mathbb{CP}^1$]

Algebraic horizontal sections classified by Schwarz

$\rightsquigarrow$ List with 15 entries:

1 dihedral family
2 Tetrahedral solutions
2 Octahedral solutions
10 Icosahedral solutions

[Rigid]

régulier. D'ailleurs, si l'intégrale générale de l'une des équations $\mathcal{E}(\lambda, \mu, \nu)$, $\mathcal{E}(1-\lambda, 1-\mu, \nu)$, $\mathcal{E}(\lambda, 1-\mu, 1-\nu)$, $\mathcal{E}(1-\lambda, \mu, 1-\nu)$ est une fonction algébrique de x, il en est évidemment de même des trois autres (nº 35).

Soient $\lambda'\pi$, $\mu'\pi$, $\nu'\pi$ les angles de celui des quatre triangles PQR, PQ'R, QP'R, P'Q'R pour lequel la somme des angles est la plus petite, les nombres λ', μ', ν' étant rangés par ordre de grandeur décroissante. Pour que l'intégrale générale de $E(\alpha, \beta, \gamma)$ soit une fonction algébrique, il faut et il suffit que les nombres λ', μ', ν', qui se déduisent de α, β, γ comme il a été expliqué, figurent dans le tableau ci-dessous de Schwarz :

	λ'	μ'	ν'	
(I)	$\frac{1}{2}$	$\frac{1}{2}$	»	double pyramide
(II)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	Tétraèdre
(III)	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	
(IV)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	Cube et octaèdre
(V)	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	
(VI)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	Icosaèdre et dodécaèdre
(VII)	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	
(VIII)	$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	
(IX)	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{5}$	
(X)	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	
(XI)	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	
(XII)	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	
(XIII)	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	
(XIV)	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	
(XV)	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	

régulier. D'ailleurs, si l'intégrale générale de l'une des équations $\mathcal{E}(\lambda, \mu, \nu)$, $\mathcal{E}(1-\lambda, 1-\mu, \nu)$, $\mathcal{E}(\lambda, 1-\mu, 1-\nu)$, $\mathcal{E}(1-\lambda, \mu, 1-\nu)$ est une fonction algébrique de x, il en est évidemment de même des trois autres (n° 35).

Soient $\lambda'\pi$, $\mu'\pi$, $\nu'\pi$ les angles de celui des quatre triangles PQR, PQ'R, QP'R, P'Q'R pour lequel la somme des angles est la plus petite, les nombres λ', μ', ν' étant rangés par ordre de grandeur décroissante. Pour que l'intégrale générale de $E(\alpha, \beta, \gamma)$ soit une fonction algébrique, il faut et il suffit que les nombres λ', μ', ν', qui se déduisent de α, β, γ comme il a été expliqué, figurent dans le tableau ci-dessous de Schwarz :

	λ'	μ'	ν'		
(I)	$\frac{1}{2}$	$\frac{1}{2}$	»	double pyramide	
(II)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	Tétraèdre	
(III)	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	Tétraèdre	
(IV)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	Cube et octaèdre	
(V)	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	Cube et octaèdre	
(VI)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	Icosaèdre et dodécaèdre	a b c
(VII)	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	Icosaèdre et dodécaèdre	b b d
(VIII)	$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	Icosaèdre et dodécaèdre	b c c
(IX)	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{5}$	Icosaèdre et dodécaèdre	a c d
(X)	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	Icosaèdre et dodécaèdre	b c d
(XI)	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	Icosaèdre et dodécaèdre	d d d
(XII)	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	Icosaèdre et dodécaèdre	b b c
(XIII)	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	Icosaèdre et dodécaèdre	c c c
(XIV)	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	Icosaèdre et dodécaèdre	a b d
(XV)	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	Icosaèdre et dodécaèdre	b c d

A_5 conjugacy classes

a ½-turn
b ⅓-
c ⅕-
$d=c^2$ ⅖-

Naive generalisations

One more pole :- $\sum_1^3 \frac{A_i}{z-a_i} dz$ WLOG $a_1, a_2, a_3 = 0, t, 1$

A $A_i \in sl_2(\mathbb{C})$

B A_i 3×3 rank 1

[Both minimally non-rigid – 2d moduli spaces]

Qn Analogue of Schwarz's list for these?

– can now answer this "nonrigid Schwarz list"

– still linear

Example of problem B:

Full symmetry group - icosahedral reflection group (order 120)

$$H = \left\langle r_1, r_2, r_3 \;\middle|\; \begin{array}{c} r_1^2 = r_2^2 = r_3^2 = 1 \\ (r_1 r_2)^2 = (r_2 r_3)^3 = (r_3 r_1)^5 = 1 \end{array} \right\rangle$$

$$\subset O_3(\mathbb{R}) \subset GL_3(\mathbb{C})$$

- look for connections on rank 3 bundles / $\mathbb{P}^1 \setminus 4$ points with monodromy H (generated by 3 reflections)

- (essentially) solved around 1997 by Dubrovin - Mazzocco

  3 inequivalent triples of generating reflections

  1 ~ K. Saito's icosahedral Frobenius manifold

  1 involves 10 pages of 40 digit integers

$$
\begin{aligned}
&-1226684412907984419281022032089194096771900\,t^8\\
&-1114701349894370233505605371103641055314707\,t^9\\
&+7066981488325984858331373729957287460068888\,t^{10}\\
&-2308855972786750597680740934867334499829866\,t^{11}\\
&+401107602137819663065957554245914269524 08\,t^{12}\\
&-294440693873880801923448428244117399261 3\,t^{13}\\
&+29909989810256194655311832623132956\,t^{14})\ x\ (1+x)\ y^{15}\\
&+3\ (-19345311524103689299806429866595584344434933\\
&+1658808400180625178945241486611794108530725 46\,t\\
&-43397535118666152789919051041986103168157 7223\,t^2\\
&+5155163066743090517140960864920723318089180 60\,t^3\\
&-2835628767616075959790243437839552709908522 89\,t^4\\
&+3508971787065203716652886578207124228491873 4\,t^5\\
&+3329792899012718704983130445738794368757890 9\,t^6\\
&-1291776424485166487282762047255608280322685 6\,t^7\\
&-2667136232453283569559792524882581432924 63\,t^8\\
&+5559001988444403518149870305222631626523 34\,t^9\\
&+3448091251995758234969231253855658313155 95\,t^{10}\\
&-3256890724598070084571219080753719914837 16\,t^{11}\\
&+1173884397830202068948971444600708463329 49\,t^{12}\\
&-2112368807268636856817019649675393743718 2\,t^{13}\\
&+15691615887424347602822354800901000822 55\,t^{14})\ x\,y^{16}\\
&+3\ (97832997604889480302194330060835702966893 57\\
&-5932111934791854365993067652198438404216943 0\,t\\
&+1414164778375296517266862645727728221934300 55\,t^2\\
&-1770968098782894567939037963774764552576735 00\,t^3\\
&+1279075864796514223185644108359081927867633 65\,t^4\\
&-5437265830913964073343929602104804972674669 8\,t^5\\
&+1348839437598325917838626903107782654132367 9\,t^6
\end{aligned}
$$

Dubrovin – Mazzocco
Preprint ~ 1997

Nonlinear analogue — Painlevé VI equation

Explicit form of simplest (abelian) Gauss-Manin systems are Gauss hypergeometric equation

[periods of families of elliptic curves - Gauss]

Explicit form of simplest non-abelian Gauss-Manin connection is the Painlevé VI equation

- "nonlinear" analogue of hgeom. equation
- solutions branch at $0, 1, \infty \in \mathbb{P}^1$ (still)

Main question Analogue of Schwarz's list for P_{VI}?

C

- still open
- will describe what is known + methods used

What comes after A_5?

will see various answers:

$$\begin{array}{ccccc} A_5 & = & PSL_2(5) & = & \Delta_{235} \\ \rightsquigarrow & & \rightsquigarrow & & \rightsquigarrow \\ A_6 & & PSL_2(7) & & \Delta_{237} \end{array}$$

[Also, next Coxeter group after $H = H_3$ is F_4 which appears too]

What is Painlevé $\underline{\text{VI}}$?

- explicit form of simplest non-Abelian Gauss–Manin connection

- equation controlling isomonodromy deformations of certain log connections/Fuchsian systems on $\mathbb{P}^1$

- most general 2nd order ODE with Painlevé property

- certain dimensional reduction of ASDYM equations

- equation related to certain elliptic integrals with moving endpoints (R. Fuchs/Manin)

The Painlevé VI equation (P_{VI}):

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{(t-1)}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

where the constants $\alpha, \beta, \gamma, \delta$ are related to the parameters $(\theta_1, \theta_2, \theta_3, \theta_4)$ by:

$$\alpha = (\theta_4 - 1)^2/2, \qquad \beta = -\theta_1^2/2, \qquad \gamma = \theta_3^2/2, \qquad \delta = (1 - \theta_2^2)/2.$$

The Painlevé VI equation (P_{VI}):

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{(t-1)}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right)$$

where the constants $\alpha, \beta, \gamma, \delta$ are related to the parameters $(\theta_1, \theta_2, \theta_3, \theta_4)$ by:

$$\alpha = (\theta_4 - 1)^2/2, \qquad \beta = -\theta_1^2/2, \qquad \gamma = \theta_3^2/2, \qquad \delta = (1-\theta_2^2)/2.$$

Main properties (well-known)

- Painlevé property – critical singularities at $0, 1, \infty$:
  "Any local solution $y(t)$ near $t \in \mathbb{P}^1 \setminus \{0,1,\infty\}$ extends to meromorphic function on universal cover"

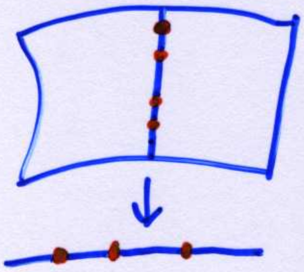
- Trichotomy
  Any solution is either
  - a 'new' transcendental function
  - a solution of a 1st order Riccati eqn
  - an algebraic function

- Governs isomonodromic deformations of rank 2 log. connections $\sum_1^3 \frac{A_i}{z-a_i}$ on $\mathbb{P}^1$ (type A)
  Eigenvalues $A_i = \pm\theta_i/2$ $\quad (A_4 = -\sum_1^3 A_i)$

- $W_{aff}(F_4)$ symmetry group (standard action on $\mathbb{C}^4 \ni (\theta_1, \theta_2, \theta_3, \theta_4)$

Conceptual approach

Consider universal family of $\mathbb{P}^1$'s with 4 punctures (ordered)



$$\begin{array}{ccc} C & \hookleftarrow & \mathcal{F} \cong \mathbb{P}^1 \setminus 4 \text{ points} \\ \downarrow & & \\ B := \mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0,1,\infty\} & & \end{array}$$

- Replace each fibre $\mathcal{F}$ by $H^1(\mathcal{F}, G)$, $G = SL_2(\mathbb{C})$

Two viewpoints here on H^1 :-

Betti   Moduli of π_1 representations $\mathrm{Hom}(\pi_1(\mathcal{F}), G)/G$

$\Uparrow \wr$ Riemann-Hilbert

DeRham   Moduli of connections on holomorphic vector bundles

- Get two (nonlinear) fibrations over $B = \mathcal{M}_{0,4}$
- As in abelian case get flat connection on bundle (now nonlinear connection)

$$\begin{tikzcd} \mathcal{M}_{DR} \arrow[r, "RH"] \arrow[d] & \mathcal{M}_{Betti} \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Two descriptions of connection:

- Betti (periods $\rightsquigarrow$ monodromy)

  - keep monodromy representation constant

- DeRham $\left(\begin{array}{l} \text{one forms} \rightsquigarrow \text{connections on vector bundles} \\ \text{closed} \rightsquigarrow \text{flat} \end{array}\right)$

  - extend flat connection on fibre $\mathcal{Y}$ to full flat connection on family of fibres & restrict to another fibre

Applications: DR $\rightsquigarrow$ explicit nonlinear equations $\rightsquigarrow$ P_{VI}

Betti $\rightsquigarrow$ explicit description of monodromy of nonlinear connection

Explicit nonlinear equations

$\mathcal{M}_{DR}$ well approximated by moduli of log. connections on trivial bundles over $\mathbb{P}^1$:-

$$\mathcal{M}^* \cong \left\{ d - \sum_1^3 \frac{A_i}{z-a_i} dz \right\} / \text{isomorphism}$$

$$\cong \left\{ (A_1, \ldots, A_4) \,\middle|\, A_i \in \mathfrak{g},\ \sum_1^4 A_i = 0 \right\} / G \quad \times B$$

Nonlinear connection on $\begin{matrix}\mathcal{M}^* \\ \downarrow \\ B\end{matrix}$ was computed by Schlesinger:

Horizontal sections satisfy
$$\frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{a_i - a_j} \qquad i \neq j$$

~ flatness of full connection $\quad d - \sum A_i \dfrac{d(z-a_i)}{z-a_i}$

Note
- fibres of $\mathcal{M}^*/B$ 6d Poisson manifolds
- Schlesinger's equations preserve the eigenvalues of each A_i $(i = 1,2,3,4)$
- flows restrict to 2d symplectic leaves
- choose coords x, y on leaves $\Rightarrow$ coupled 1st order ODEs
- eliminate $x \Rightarrow$ 2nd order ode for $y(t)$ – Painlevé $\overline{\underline{\text{VI}}}$ (t = coord on $B = \mathcal{M}_{0,4}$)

Monodromy of Painlevé VI

$\sim$ monodromy of connection on $\begin{array}{c} M_{Betti} \\ \downarrow \\ B \end{array}$

- connection is complete & flat so $\Leftrightarrow$

action of $\pi_1(B) \cong F_2 \circlearrowleft$ fibre M_t

$$\begin{array}{c} \shortparallel \\ \mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus 4\,\text{points}), G)/G \end{array}$$

Given choice of loops generating $\pi_1(\mathbb{P}^1 \setminus 4\,\text{points})$ get

$$M_t \cong \{(M_1, M_2 M_3 M_4) \mid M_i \in G,\ M_4 M_3 M_2 M_1 = 1\}/G$$
$$\cong G^3/G$$

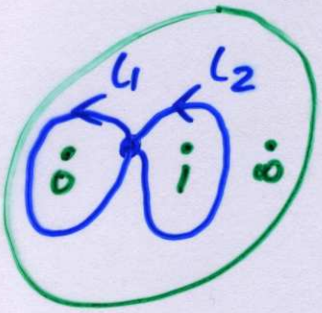
– universal family of cubic surfaces (Fricke-Klein/Cayley)

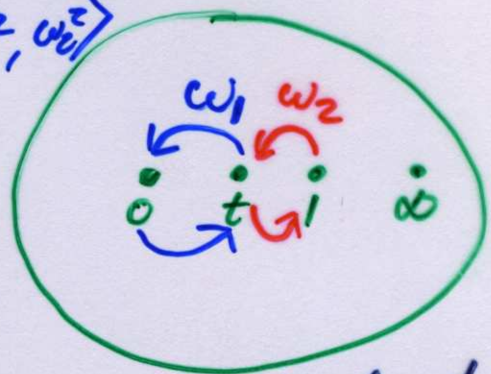
What is the monodromy action $\pi_1(B) \circlearrowright M_t$?

$$\pi_1(B) \cong F_2 \cong \text{Pure mapping class group of } (\mathbb{P}^1, 0, t, 1, \infty)$$

$$\pi_1(B) = \langle l_1, l_2 \rangle \xrightarrow{\sim} \langle \omega_1^2, \omega_2^2 \rangle = \text{Pure mapping class group}$$

$$(l_i \mapsto \omega_i^2)$$





Dehn twists

- Mapping class gp acts naturally on $M_t = \mathrm{Hom}_{\underline{C}}(\pi_1(\mathbb{P}^1 \setminus \{0,t,1,\infty\}), G)/G$

by "pushing forward loops"

$$\left[f(\rho)(\gamma) = \rho(f \circ \gamma) \qquad \begin{array}{l} \rho \in M_t \\ \gamma \in \pi_1 \\ f : \mathbb{P}^1 \to \mathbb{P}^1 \text{ diffeo} \end{array} \right]$$

- This is the monodromy action of $\boxed{\pi_1}$

- Explicitly on monod. matrices:

$$\omega_1(M_1, M_2, M_3) = (M_2, M_2 M_1 M_2^{-1}, M_3)$$

$$\omega_2(\text{———}) = (M_1, M_3, M_3 M_2 M_3^{-1})$$

Definition

An algebraic solution to P_{VI} is an irreducible polynomial $F(y,t) \in \mathbb{C}[y,t]$ s.t the algebraic function $y(t)$ defined implicitly by

$$F(y(t),t) = 0$$

solves P_{VI} for some value of the parameters

Definition'

... is an (irreducible compact) algebraic curve Π and two rational functions $y, t : \Pi \to \mathbb{P}^1$ s.t

① t is a Belyi map (branch locus $\subset \{0,1,\infty\}$)

② $y(t)$ solves P_{VI} (for some parameters)

A $\Longrightarrow$ C

If $M_1, M_2, M_3 \in$ finite subgroup of $SL_2(\mathbb{C}) = G$
then $\mathcal{F}_2$ orbit of (M_1, M_2, M_3) finite
- so expect P_{VI} solution algebraic

finite $\mathcal{F}_2$ orbit $\Leftrightarrow$ solution has finite no. branches

- {branches of solution} $\cong$ $\mathcal{F}_2$ orbit of conjugacy classes of triples (M_1, M_2, M_3)
- $\mathcal{F}_2$ action on orbit ~ permutation representation of Belyi cover

$$\begin{array}{c} \Pi \\ \downarrow t \\ \mathbb{P}^1 \end{array}$$

Basic examples of algebraic solutions to Painlevé VI (Hitchin, Dubrovin):

Three-branch tetrahedral solution:

$$y = \frac{(s-1)(s+2)}{s(s+1)}, \qquad t = \frac{(s-1)^2(s+2)}{(s+1)^2(s-2)}$$

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (2/3, 1/3, 1/3, 2/3)$$

Four-branch dihedral solution:

$$y = \frac{s^2(s+2)}{s^2+s+1}, \qquad t = \frac{s^3(s+2)}{2s+1}$$

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/2, 1/2, 1/2)$$

Four-branch octahedral solution:

$$y = \frac{(s-1)^2}{s(s-2)}, \qquad t = \frac{(s+1)(s-1)^3}{s^3(s-2)}$$

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/4, 1/4, 1/4, 1/4)$$

Elliptic dihedral solution
Hitchin 1996
12 branches, genus 1

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/2, 1/2, 1/2)$$

$$y = \frac{(3\,s-1)\,(s^2-4\,s-1)\,(s^2+u)\,(s\,(s+2)-u)}{(3\,s^3+7\,s^2+s+1)\,(s^2-u)\,(s\,(s-2)+u)}$$

$$t = \frac{(s^2+u)^2\,(s\,(s+2)-u)\,(s\,(s-2)-u)}{(s^2-u)^2\,(s\,(s+2)+u)\,(s\,(s-2)+u)}$$

where s, u satisfy:

$$u^2 = s\,(s^2+s-1)$$

Beyond Platonic P_{VI} solutions

$$B \dashrightarrow C$$

Theorem (– '02)

Painlevé VI also controls isomonodromic deformations of type B connections $\sum_1^3 \frac{B_i}{z-a_i}dz$ (B_i 3×3 rank 1)

– similar to sl_2 argument (2d moduli spaces)

– as before: finite monodromy groups $\rightsquigarrow$ algebraic solutions (one expects)

Look at finite subgroups $Gl_3(\mathbb{C})$ generated by three complex reflections ("id + rank 1")

–classified by Shephard–Todd (1954)

If $y(t)$ solves $\mathrm{P_{VI}}$ with parameters

$$\theta_1 = \lambda_1 - \mu_1, \quad \theta_2 = \lambda_2 - \mu_1, \quad \theta_3 = \lambda_3 - \mu_1, \quad \theta_4 = \mu_3 - \mu_2$$

and we define $x(t)$ via

$$x = \frac{1}{2}\left(\frac{(t-1)y' - \theta_1}{y} + \frac{y' - 1 - \theta_2}{y - t} - \frac{t\,y' + \theta_3}{y - 1}\right)$$

then the family of logarithmic connections

$$d - \left(\frac{B_1}{z} + \frac{B_2}{z-t} + \frac{B_3}{z-1}\right) dz$$

will be isomonodromic as t varies, where

$$B_1 = \begin{pmatrix} \lambda_1 & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & \lambda_2 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & \lambda_3 \end{pmatrix}$$

$$\begin{aligned} b_{12} &= \lambda_1 \;\; - \mu_3 y + (\mu_1 - xy)(y-1), & b_{32} &= (\mu_2 - \lambda_2 - b_{12})/t, \\ b_{13} &= \lambda_1 t - \mu_3 y + (\mu_1 - xy)(y-t), & b_{23} &= (\mu_2 - \lambda_3)t - b_{13}, \\ b_{21} &= \lambda_2 + \frac{\mu_3(y-t) - \mu_1(y-1) + x(y-t)(y-1)}{t-1}, & b_{31} &= (\mu_2 - \lambda_1 - b_{21})/t. \end{aligned}$$

(Triply generated) 3d complex reflection groups

Dihedral

Tetrahedral

Octahedral

Icosahedral

$G(m, p, 3)$

Klein $\qquad 2\,PSL_2(7)$

Hesse 1

Hesse 2

Valentiner $\qquad 6\,A_6$

Construction problem

triple of generating reflections $\rightsquigarrow$ finite $\mathcal{F}_2$ orbit
$\rightsquigarrow$ Painlevé curve Π topologically

- need Π explicitly & function y on it

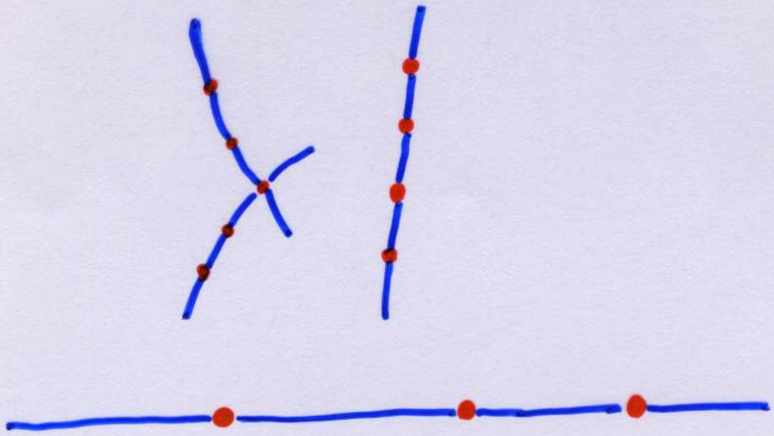
e-g Klein $\rightsquigarrow$ Π of genus 0 & $\deg(t) = 7$

Valentiner $\rightsquigarrow$ 3 Painlevé curves Π
- each of genus 1
- $\deg(t) = 15, 15, 24$

Key inputs

1. Jimbo
   - leading asymptotics of P_{VI} solution y in terms of linear monodromy (M_1, M_2, M_3)

Degenerate to stable curve & solve Riemann-Hilbert problems there:



$\overline{B} = \overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$

Theorem. (Jimbo 1982)

Suppose we have four matrices $M_j \in \mathrm{SL}_2(\mathbb{C})$, $j = 1, 2, 3, 4$ satisfying

a) $M_4M_3M_2M_1 = 1$,

b) M_j has eigenvalues $\{\exp(\pm\pi i\theta_j)\}$ with $\theta_j \notin \mathbb{Z}$,

c) $\mathrm{Tr}(M_1M_2) = 2\cos(\pi\sigma)$ for some nonzero $\sigma \in \mathbb{C}$ with $0 \le \mathrm{Re}(\sigma) < 1$,

d) None of the eight numbers

$$\theta_1 \pm \theta_2 \pm \sigma, \quad \theta_1 \pm \theta_2 \mp \sigma, \quad \theta_4 \pm \theta_3 \pm \sigma, \quad \theta_4 \pm \theta_3 \mp \sigma$$

is an even integer.

Then the leading term in the asymptotic expansion at zero of the corresponding Painlevé VI solution $y(t)$ on the branch corresponding to $[(M_1, M_2, M_3)]$ is

$$\left(\frac{(\theta_1+\theta_2+\sigma)(-\theta_1+\theta_2+\sigma)(\theta_4+\theta_3+\sigma)}{4\sigma^2(\theta_4+\theta_3-\sigma)\widehat{s}}\right) t^{1-\sigma}$$

where

$$\widehat{s} = c \times s, \qquad s = \frac{a+b}{d}$$

$$a = e^{\pi i\sigma}(i\sin(\pi\sigma)\cos(\pi\sigma_{23}) - \cos(\pi\theta_2)\cos(\pi\theta_4) - \cos(\pi\theta_1)\cos(\pi\theta_3))$$

$$b = i\sin(\pi\sigma)\cos(\pi\sigma_{13}) + \cos(\pi\theta_2)\cos(\pi\theta_3) + \cos(\pi\theta_4)\cos(\pi\theta_1)$$

$$d = 4\sin\left(\frac{\pi}{2}(\theta_1+\theta_2-\sigma)\right)\sin\left(\frac{\pi}{2}(\theta_1-\theta_2+\sigma)\right)\sin\left(\frac{\pi}{2}(\theta_4+\theta_3-\sigma)\right)\sin\left(\frac{\pi}{2}(\theta_4-\theta_3+\sigma)\right)$$

$$c = \frac{(\Gamma(1-\sigma))^2\,\widehat{\Gamma}(\theta_1+\theta_2+\sigma)\,\widehat{\Gamma}(-\theta_1+\theta_2+\sigma)\,\widehat{\Gamma}(\theta_4+\theta_3+\sigma)\,\widehat{\Gamma}(-\theta_4+\theta_3+\sigma)}{(\Gamma(1+\sigma))^2\,\widehat{\Gamma}(\theta_1+\theta_2-\sigma)\,\widehat{\Gamma}(-\theta_1+\theta_2-\sigma)\,\widehat{\Gamma}(\theta_4+\theta_3-\sigma)\,\widehat{\Gamma}(-\theta_4+\theta_3-\sigma)}$$

where $\widehat{\Gamma}(x) := \Gamma(\frac{1}{2}x+1)$ (with Γ being the usual gamma function) and where $\sigma_{jk} \in \mathbb{C}$ $(j, k \in \{1, 2, 3\})$ is determined by $\mathrm{Tr}(M_jM_k) = 2\cos(\pi\sigma_{jk}), 0 \le \mathrm{Re}(\sigma_{jk}) \le 1$, so $\sigma = \sigma_{12}$.

(2) Relate systems (A) & (B) on both DeRham & Betti sides

- monodromy changes in highly non-trivial way:

Klein reflection group $\rightsquigarrow$ Δ_{237}

Valentiner group $\rightsquigarrow$ A_5

- apparently procedure is complex analytic version of N. Katz's "middle convolution functor"

Sketch

Fix distinct $a_1,\dots,a_n\in\mathbb{C}$

Let $A_0=\mathrm{diag}(a_1,\dots,a_n)$

$$\begin{tikzcd}
& A\in\mathfrak{gl}_n(\mathbb{C}) \arrow[dl, "A_i=E_iA"] \arrow[dr] & \\
d-\sum_1^n \frac{A_i}{z-a_i}dz \arrow[rr, leftrightarrow, "\text{Fourier-Laplace}"'] \arrow[d, "\text{Riemann-Hilbert}"] & & d-\left(\frac{A_0}{z^2}+\frac{A}{z}\right)dz \arrow[d, "\text{Stokes}"'] \\
\begin{matrix} r_1,\dots,r_n \\ r_i=1+e_i\otimes\alpha_i \end{matrix} \arrow[rr, leftrightarrow, "\text{Killing-Coxeter}", "(\alpha_i(e_j))=tu_+-u_-"'] \arrow[dr, "r_n\cdots r_1"'] & & u_+,u_-,t\in U_+\times U_-\times T \arrow[dl, "u_-^{-1}tu_+"] \\
& G &
\end{tikzcd}$$

Scalar shift $A\longmapsto A+\lambda I$

- tensor by $\lambda\frac{dz}{z}$ on RHS
- nontrivial convolution on LHS

$n=3$: choose λ s.t $A+\lambda$ rank $2\Rightarrow$ reducible on LHS

- take 2×2 quotient connection $\rightsquigarrow SL_2$ connection

Klein solution

seven branches

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (2/7, 2/7, 2/7, 4/7)$$

$$y = -\frac{(5\,s^2 - 8\,s + 5)\,(7\,s^2 - 7\,s + 4)}{s\,(s-2)\,(s+1)\,(2\,s-1)\,(4\,s^2 - 7\,s + 7)}$$

$$t = \frac{\left(7\,s^2 - 7\,s + 4\right)^2}{s^3\,(4\,s^2 - 7\,s + 7)^2}.$$

Corollary

For any s such that $t(s) \neq 0, 1, \infty$ the family of Fuchsian systems

$$\frac{d}{dz} - \left(\frac{B_1}{z} + \frac{B_2}{z - t(s)} + \frac{B_3}{z-1}\right)$$

has monodromy isomorphic to the Klein complex reflection group, where

$$B_1 = \begin{pmatrix} \frac{1}{2} & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & \frac{1}{2} & b_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & \frac{1}{2} \end{pmatrix}$$

$$b_{12} = \frac{14\,s^3 - 21\,s^2 + 24\,s - 22}{21 s\,(4\,s^2 - 7\,s + 7)}, \qquad b_{13} = \frac{22\,s^3 - 24\,s^2 + 21\,s - 14}{21(7\,s^2 - 7\,s + 4)},$$

$$b_{21} = \frac{14\,s^3 - 21\,s^2 + 24\,s + 5}{21\,(s-1)\,(4\,s^2 - s + 4)}, \qquad b_{23} = \frac{22\,s^3 - 42\,s^2 + 39\,s - 5}{21(7\,s^2 - 7\,s + 4)},$$

$$b_{31} = \frac{14 - 21\,s + 24\,s^2 + 5\,s^3}{21\,(s-1)\,(4\,s^2 - s + 4)}, \qquad b_{32} = \frac{22 - 42\,s + 39\,s^2 - 5\,s^3}{21 s\,(4\,s^2 - 7\,s + 7)}.$$

Icosahedral solutions with $\leqslant 4$ branches

	Degree	Genus	Walls	A_5 Type	Alcove Point	No.	Group (size)
1	1	0	1	$a\,b\,c$	31, 19, 11, 1	192	1
2	1	0	1	$a\,b\,d$	37, 17, 13, 7	192	1
3	1	0	1	$a\,c\,d$	33, 21, 9, 3	192	1
4	1	0	1	$b\,c\,d$	28, 16, 8, 4	192	1
5	1	0	2	$b^2\,c$	26, 14, 6, 6	96	1
6	1	0	2	$b^2\,d$	38, 18, 18, 2	96	1
7	1	0	2	$b\,c^2$	22, 10, 10, 2	96	1
8	1	0	2	$b\,d^2$	34, 14, 10, 10	96	1
9	1	0	3	c^3	18, 6, 6, 6	32	1
10	1	0	3	d^3	42, 18, 18, 6	32	1
11	2	0	2	$b^2\,c^2$	42, 18, 10, 10	96	2
12	2	0	2	$b^2\,d^2$	50, 10, 6, 6	96	2
13	2	0	2	$c^2\,d^2$	42, 18, 6, 6	96	2
14	3	0	1	$b\,c^2\,d$	40, 16, 8, 8	288	S_3
15	3	0	1	$b\,c\,d^2$	40, 8, 4, 4	288	S_3
16	4	0	2	$a\,c^3$	33, 9, 9, 9	128	A_4
17	4	0	2	$a\,d^3$	51, 3, 3, 3	128	A_4
18	4	0	2	$c^3\,d$	30, 6, 6, 6	128	A_4
19	4	0	2	$c\,d^3$	42, 6, 6, 6	128	A_4

Icosahedral solutions with $\geqslant 5$ branches

	Degree	Genus	Walls	A_5 Type	Alcove Point	No.	Group (size)	
20	5	0	1	$b^2\,c\,d$	44, 12, 12, 4	480	S_5	
21	5	0	2	$c^2\,d^2$	36, 12, 0, 0	240	S_5	Kitaev
22	6	0	1	$b\,c^2\,d$	34, 10, 2, 2	576	S_6	
23	6	0	1	$b\,c\,d^2$	46, 14, 10, 2	576	S_6	
24	8	0	1	$a\,c^2\,d$	39, 15, 3, 3	768	A_8	
25	8	0	1	$a\,c\,d^2$	45, 9, 9, 3	768	A_8	
26	9	1	2	$b\,c^3$	28, 4, 4, 4	288	A_9	Kitaev
27	9	1	2	$b\,d^3$	52, 8, 8, 4	288	A_9	
28	10	0	2	$a^2\,c\,d$	48, 12, 6, 6	480	$2^7\,3\,5$	
29	10	0	2	$b^3\,c$	46, 14, 14, 6	320	A_{10}	
30	10	0	2	$b^3\,d$	42, 2, 2, 2	320	A_{10}	
31	10	0	3	c^4	24, 0, 0, 0	80	A_{10}	DM
32	10	0	3	d^4	48, 0, 0, 0	80	A_{10}	
33	12	0	0	$a\,b\,c\,d$	43, 11, 7, 1	2304	A_{12}	
34	12	1	1	$a\,b\,c^2$	37, 13, 5, 5	1152	A_{12}	
35	12	1	1	$a\,b\,d^2$	49, 5, 5, 1	1152	A_{12}	
36	12	1	1	$b^2\,c\,d$	38, 6, 6, 2	1152	$2^9\,3^2\,5$	
37	15	1	2	$b^3\,c$	36, 4, 4, 4	480	A_{15}	Valentiner
38	15	1	2	$b^3\,d$	48, 8, 8, 8	480	A_{15}	
39	15	1	2	$b^2\,c^2$	32, 8, 0, 0	720	S_{15}	
40	15	1	2	$b^2\,d^2$	44, 4, 0, 0	720	S_{15}	
41	18	1	3	b^4	40, 0, 0, 0	144	$2^{14}\,3^4\,5\,7$	DM
42	20	1	1	$a\,b^2\,c$	41, 9, 9, 1	1920	A_{20}	
43	20	1	1	$a\,b^2\,d$	47, 7, 3, 3	1920	A_{20}	
44	20	1	3	$a^2\,c^2$	42, 18, 0, 0	480	$2^{17}\,3^4\,5^2\,7$	
45	20	1	3	$a^2\,d^2$	54, 6, 0, 0	480	$2^{17}\,3^4\,5^2\,7$	
46	24	1	2	$a\,b^3$	45, 5, 5, 5	768	$2^{20}\,3^5\,5^2\,7\,11$	Valentiner
47	30	2	2	$a^2\,b\,c$	46, 14, 4, 4	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$	
48	30	2	2	$a^2\,b\,d$	52, 8, 2, 2	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$	
49	36	3	3	$a^2\,b^2$	50, 10, 0, 0	864	$2^{23}\,3^4\,5\,7$	
50	40	3	3	$a^3\,c$	51, 9, 9, 9	320	$2^{25}\,3^4\,5^2\,7$	
51	40	3	3	$a^3\,d$	57, 3, 3, 3	320	$2^{25}\,3^4\,5^2\,7$	
52	72	7	3	$a^3\,b$	55, 5, 5, 5	576	$2^{32}\,3^4\,5\,7$	

"Generic" solution, genus zero, 12 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (2/5, 1/2, 1/3, 4/5)$:

$$y = -\frac{9s(s^2+1)(3s-4)(15s^4-5s^3+3s^2-3s+2)}{(2s-1)^2(9s^2+4)(9s^2+3s+10)}$$

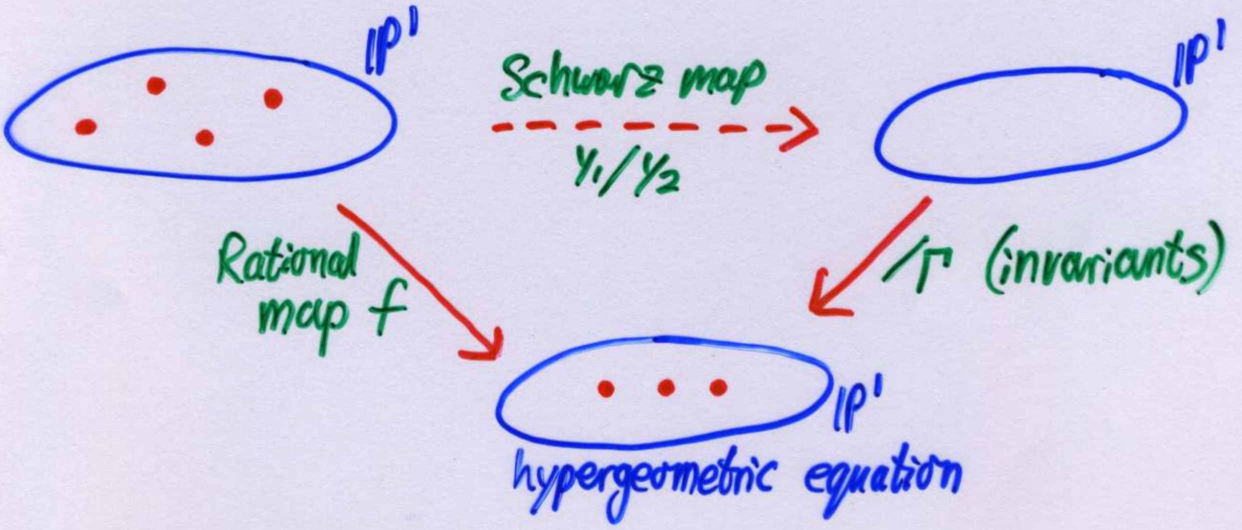
$$t = \frac{27s^5(s^2+1)^2(3s-4)^3}{4(2s-1)^3(9s^2+4)^2}$$

$$F(y,t) =$$

$$\begin{gathered}
(15524784\,t^2 - 5373216\,t + 1350000)\,y^{12} - (128381760\,t^2 - 13366080\,t)\,y^{11} + \\
(5425704\,t^3 + 496677744\,t^2 - 30539160\,t)\,y^{10} - \\
(14929920\,t^4 + 41364000\,t^3 + 866759680\,t^2 - 2928160\,t)\,y^9 + \\
(107546535\,t^4 - 508275750\,t^3 + 747613335\,t^2 - 1837080\,t)\,y^8 - \\
(24385536\,t^5 - 285548724\,t^4 - 2437066824\,t^3 + 74927724\,t^2 + 944784\,t)\,y^7 + \\
(58212000\,t^5 - 2865570750\,t^4 - 4456260900\,t^3 + 17631810\,t^2)\,y^6 - \\
(49787136\,t^6 - 904003584\,t^5 - 7215732804\,t^4 - 2130570936\,t^3 - 12872196\,t^2)\,y^5 - \\
(413500320\,t^6 + 3724484160\,t^5 + 4839581265\,t^4 + 162430110\,t^3 + 3750705\,t^2)\,y^4 + \\
(3001304640\,t^6 + 74794560\,t^5 + 2710584000\,t^4 - 380946240\,t^3)\,y^3 - \\
(940800000\,t^7 + 977540640\,t^6 - 726801696\,t^5 + 939255264\,t^4 - 72013536\,t^3)\,y^2 + \\
(1176000000\,t^7 - 1481095680\,t^6 + 765158400\,t^5)\,y - \\
(1920800000\,t^8 - 7212800000\,t^7 + 10522980864\,t^6 - 6913299456\,t^5 + 1728324864\,t^4)
\end{gathered}$$

Pullbacks (Klein, R. Fuchs, ..., Kitaev, C. Doran, ...)

Klein showed all 2nd order Fuchsian equations with finite monodromy are (essentially) pullbacks of hypergeometric equations:



so isomonodromic family of ODEs $\sim$ family of rational maps

Key observation: algebraicity of deformation comes from that of rational maps (Hurwitz spaces)
(Doran, Kitaev) not from finiteness of monodromy representation

C. Doran JDG 2001

regular singular point at λ, and precisely four non-apparent regular singular points at $\{0, 1, \infty, t\}$. The local monodromies about these points do not vary with $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. By Lemma 2.9, we thus know that λ as a function of t determines a solution to a Painlevé VI equation as described. q.e.d.

A direct application of this criterion to the natural hypergeometric local systems associated to triangles yields the following three corollaries:

Corollary 4.6. *The following is the complete list of topological types corresponding to algebraic Painlevé* VI *solutions coming from pullback from arithmetic Fuchsian triangle groups, together with the description of the corresponding triangle:*

$(2; [2], [1,1], [1,1]; 2)$	$(2, \square, \square)$
$(3; [2,1], [3], [1,1,1]; 2)$	$(2, 3, \square)$
$(4; [2,2], [3,1], [2,1,1]; 2)$	$(2, 3, \square)$
$(4; [2,2], [4], [1,1,1,1]; 2)$	$(2, 4, \square)$
$(6; [2,2,2], [3,3], [2,2,1,1]; 2)$	$(2, 3, \square)$
$(6; [2,2,2], [3,3], [3,1,1,1]; 2)$	$(2, 3, \square)$
$(10; [2,\dots,2], [3,3,3,1], [7,1,1,1]; 2)$	$(2, 3, 7)$
$(12; [2,\dots,2], [3,3,3,3], [7,2,1,1,1]; 2)$	$(2, 3, 7)$
$(12; [2,\dots,2], [3,3,3,3], [8,1,1,1,1]; 2)$	$(2, 3, 8)$
$(18; [2,\dots,2], [3,\dots,3], [7,7,1,1,1,1]; 2)$	$(2, 3, 7)$

Here $\square$ *represents any of the possible entries as listed in Theorem* 4.4.

Note that in the case of the arithmetic triangle group $\mathrm{PSL}(2, \mathbb{Z})$, with triangle $(2, 3, \infty)$, as expected we recover from this list the topological types of the Kodaira functional inveriants of our five families. In this corollary, the restriction to arithmetic Fuchsian triangle groups is for convenience only — we just wanted a finite set of triangle groups in $\mathrm{PSL}(2, \mathbb{R})$ to which to apply our criterion, and in this case they yielded a finite list of topological types. By contrast, for some triangles one can explicitly construct infinite lists of allowable topological types (unlike the previous result, the proofs of these corollaries do not produce an exhaustive list of types, merely an infinite one):

Corollary 4.7. *There are infinitely many topological types corresponding to algebraic Painlevé* VI *solutions arising by pullback from each triangle uniformized by* $\mathbb{C}$, *except for* $(3, 3, 3)$ *which has none.*

C. Doran JDG 2001

regular singular point at λ, and precisely four non-apparent regular singular points at $\{0, 1, \infty, t\}$. The local monodromies about these points do not vary with $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. By Lemma 2.9, we thus know that λ as a function of t determines a solution to a Painlevé VI equation as described. q.e.d.

A direct application of this criterion to the natural hypergeometric local systems associated to triangles yields the following three corollaries:

Ramification indices over $0, 1, \infty$

Corollary 4.6. *The following is the complete list of topological types corresponding to algebraic Painlevé* VI *solutions coming from pullback from arithmetic Fuchsian triangle groups, together with the description of the corresponding triangle:*

Degree of rational map f

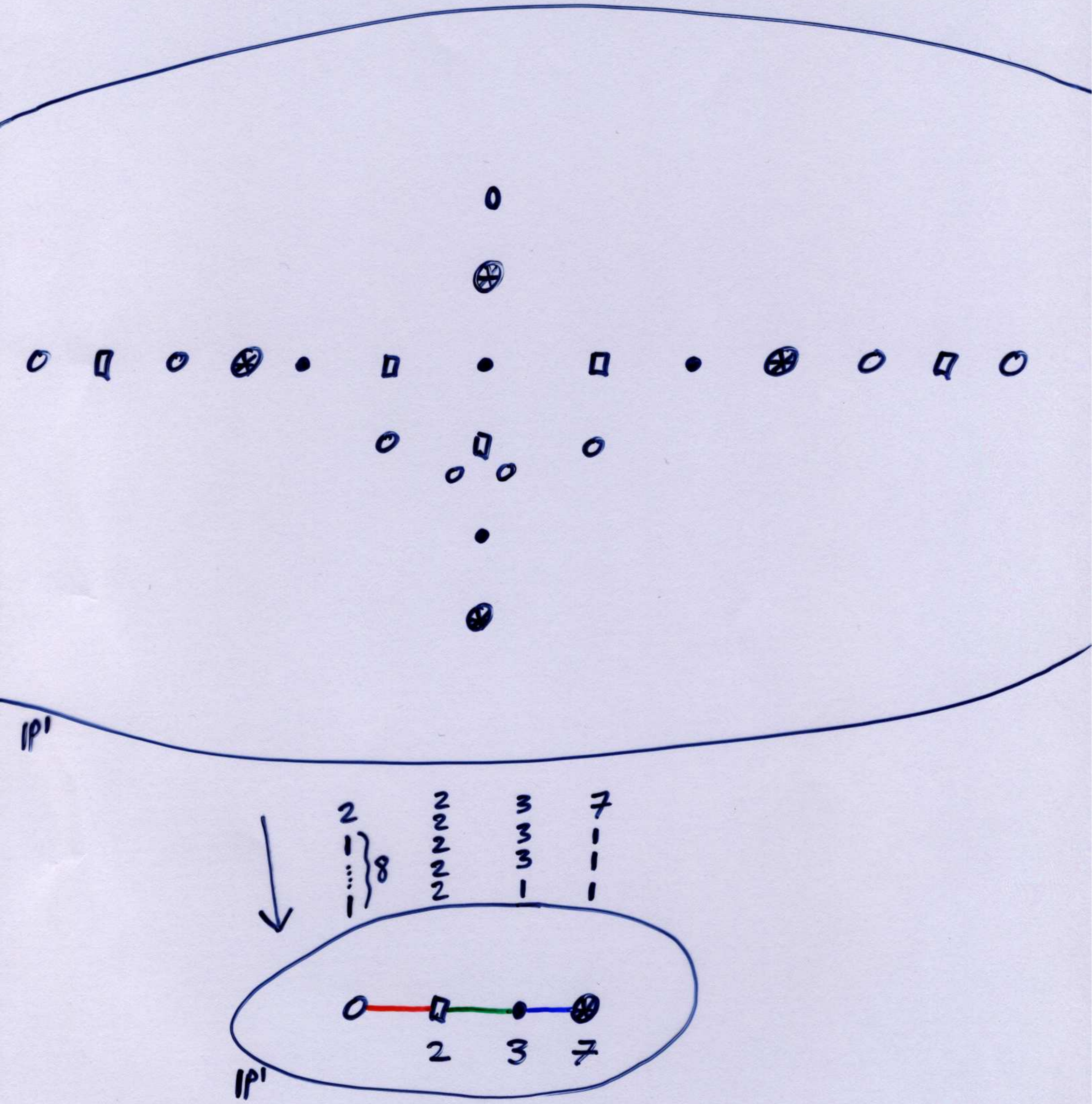
Triangle group

$(2; [2], [1,1], [1,1]; 2)$	$(2, \square, \square)$	
$(3; [2,1], [3], [1,1,1]; 2)$	$(2, 3, \square)$	
$(4; [2,2], [3,1], [2,1,1]; 2)$	$(2, 3, \square)$	≤ 4 branches
$(4; [2,2], [4], [1,1,1,1]; 2)$	$(2, 4, \square)$	
$(6; [2,2,2], [3,3], [2,2,1,1]; 2)$	$(2, 3, \square)$	
$(6; [2,2,2], [3,3], [3,1,1,1]; 2)$	$(2, 3, \square)$	
$(10; [2,\ldots,2], [3,3,3,1], [7,1,1,1]; 2)$	$(2, 3, 7)$	$g=1$, $d=18$ new
$(12; [2,\ldots,2], [3,3,3,3], [7,2,1,1,1]; 2)$	$(2, 3, 7)$	Klein
$(12; [2,\ldots,2], [3,3,3,3], [8,1,1,1,1]; 2)$	$(2, 3, 8)$	$\sqrt{t}$ or Octahedral
$(18; [2,\ldots,2], [3,\ldots,3], [7,7,1,1,1,1]; 2)$	$(2, 3, 7)$	$\sqrt{t}$

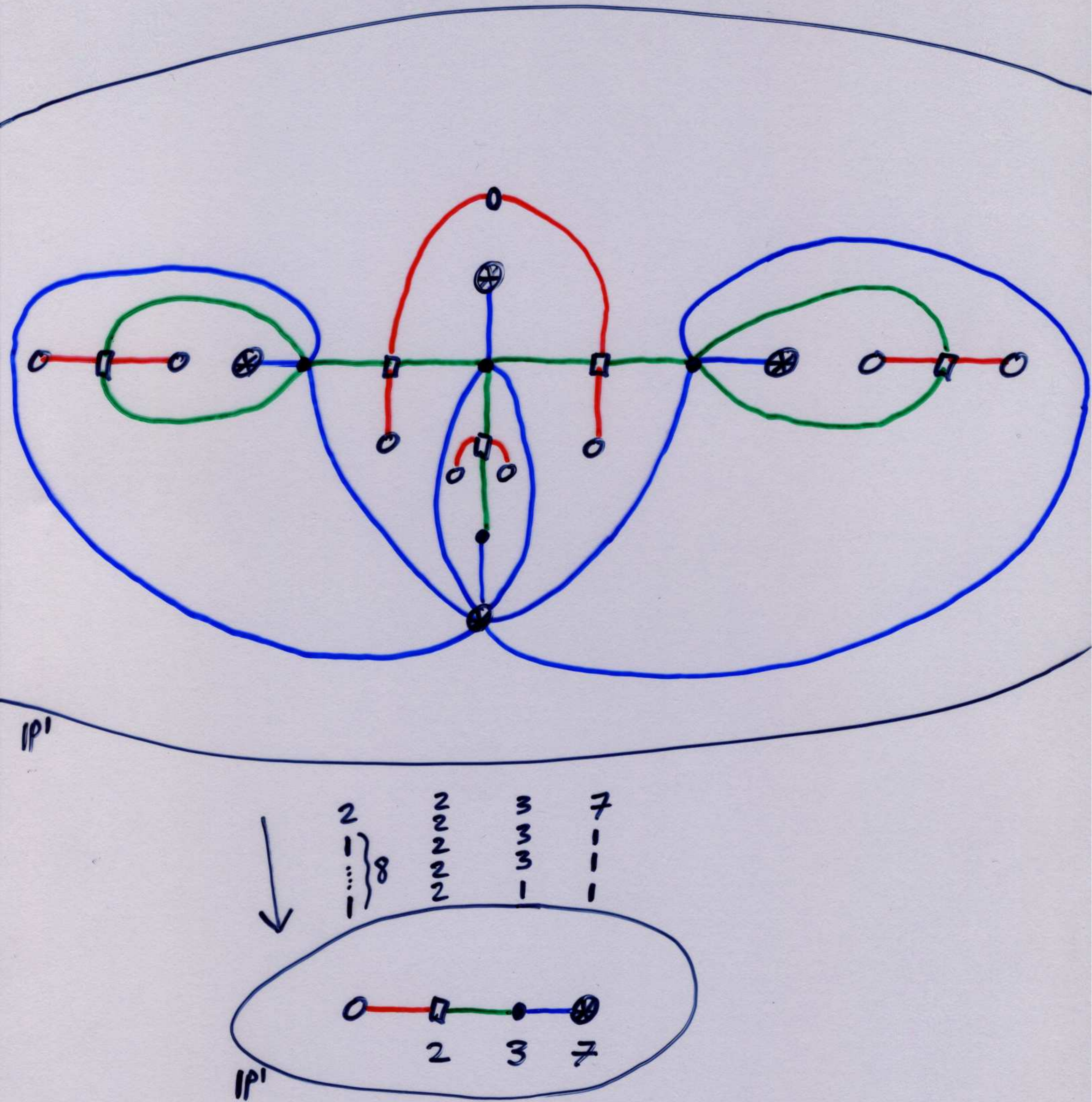
Here $\square$ *represents any of the possible entries as listed in Theorem* 4.4.

Note that in the case of the arithmetic triangle group $\mathrm{PSL}(2, \mathbb{Z})$, with triangle $(2, 3, \infty)$, as expected we recover from this list the topological types of the Kodaira functional inveriants of our five families. In this corollary, the restriction to arithmetic Fuchsian triangle groups is for convenience only — we just wanted a finite set of triangle groups in $\mathrm{PSL}(2, \mathbb{R})$ to which to apply our criterion, and in this case they yielded a finite list of topological types. By contrast, for some triangles one can explicitly construct infinite lists of allowable topological types (unlike the previous result, the proofs of these corollaries do not produce an exhaustive list of types, merely an infinite one):

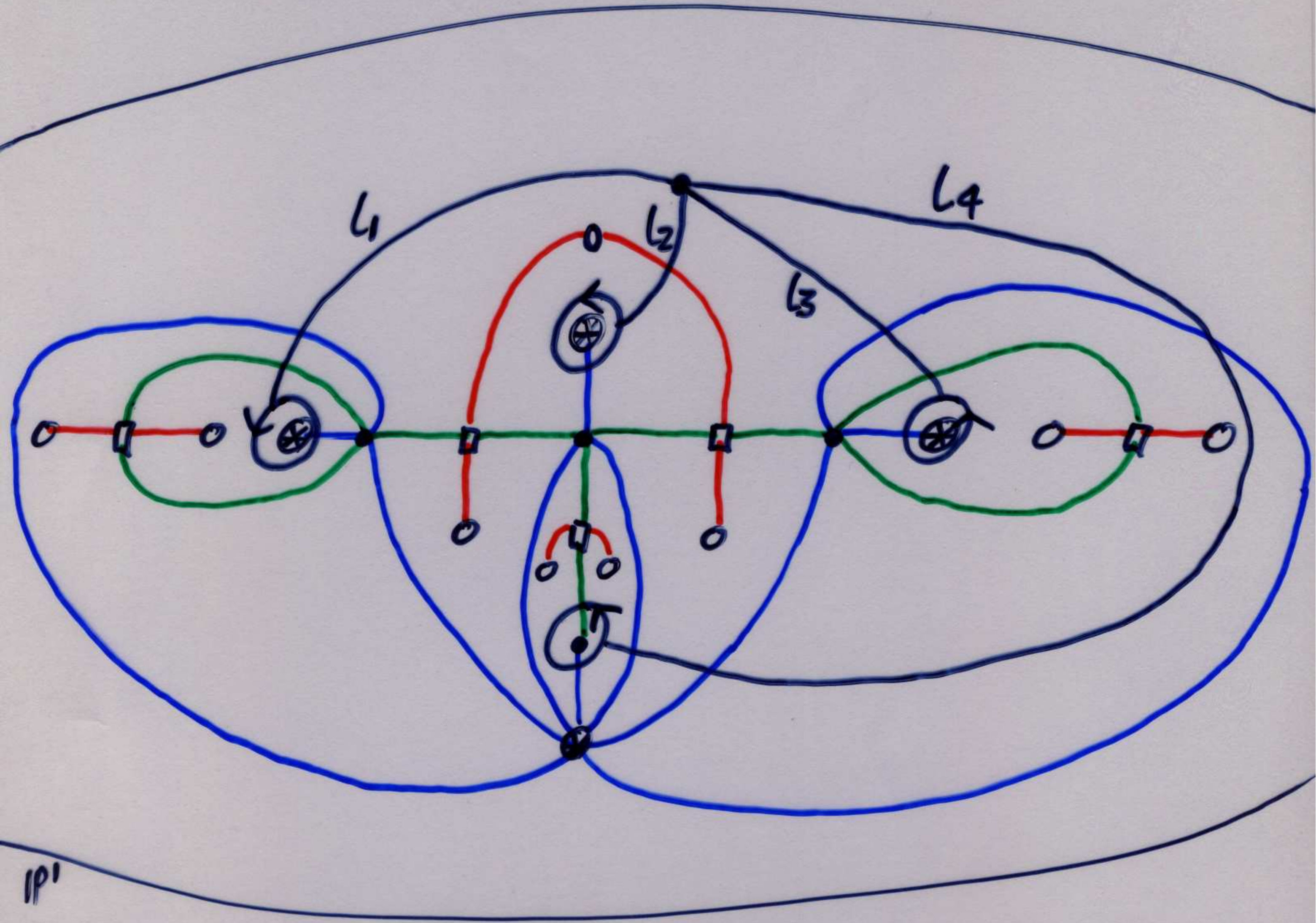
Corollary 4.7. *There are infinitely many topological types corresponding to algebraic Painlevé* VI *solutions arising by pullback from each triangle uniformized by* $\mathbb{C}$, *except for* $(3,3,3)$ *which has none.*

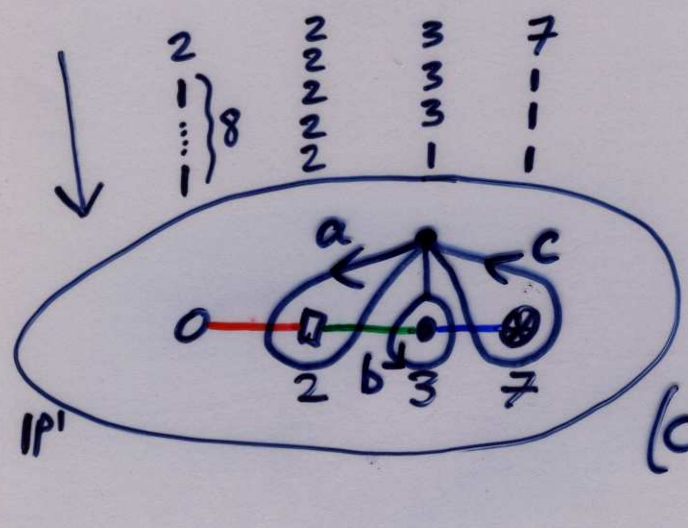


$\mathbb{P}^1$
2
1
1
8
2
2
2
2
2
3
3
3
1
7
1
1
1
2
3
7
$\mathbb{P}^1$



$\mathbb{P}^1$
2
1
⋮
1
8
2
2
2
2
2
3
3
3
1
7
1
1
1
$\mathbb{P}^1$
2
3
7





$$l_1 = c a c a^{-1} c^{-1}$$
$$l_2 = c$$
$$l_3 = c^{-1} a^{-1} c a c$$
$$l_4 = c^{-3} b c^{3}$$

$(cba = 1)$

Simple observation

Can write down topological P_{VI} solution
from topology of f, by hand
(don't need f explicitly)

- go through Doran's list & find top. solutions
- compute explicitly by previous asymptotic method.

2, 3, 7 solution

genus one, 18 branches

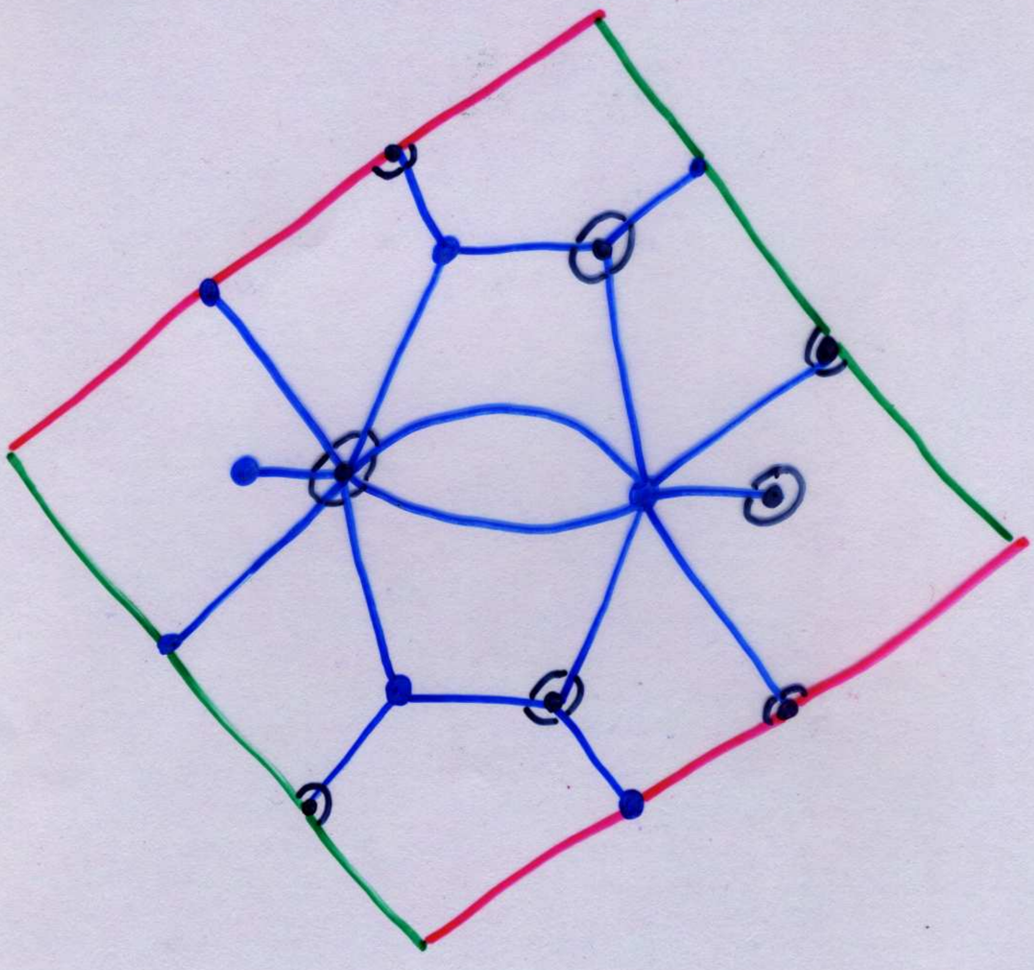
$(\theta_1, \theta_2, \theta_3, \theta_4) = (2/7, 2/7, 2/7, 1/3)$

$$y = \frac{1}{2} - \frac{\left(3\,s^8 - 2\,s^7 - 4\,s^6 - 204\,s^5 - 536\,s^4 - 1738\,s^3 - 5064\,s^2 - 4808\,s - 3199\right)u}{4\,\left(s^6 + 196\,s^3 + 189\,s^2 + 756\,s + 154\right)\left(s^2 + s + 7\right)\left(s + 1\right)}$$

$$t = \frac{1}{2} - \frac{\left(s^9 - 84\,s^6 - 378\,s^5 - 1512\,s^4 - 5208\,s^3 - 7236\,s^2 - 8127\,s - 784\right)u}{432\,s\left(s + 1\right)^2\left(s^2 + s + 7\right)^2}$$

where

$$u^2 = s\left(s^2 + s + 7\right).$$



—thanks to M. van Hoeij

Icosahedral solution 41

genus one, 18 branches

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/3, 1/3, 1/3, 1/3)$$

$$y = \frac{1}{2} - \frac{8\,s^7 - 28\,s^6 + 75\,s^5 + 31\,s^4 - 269\,s^3 + 318\,s^2 - 166\,s + 56}{18\,u\,(s-1)\,(3\,s^3 - 4\,s^2 + 4\,s + 2)}$$

$$t = \frac{1}{2} + \frac{(s+1)\left(32\,(s^8+1) - 320\,(s^7+s) + 1112\,(s^6+s^2) - 2420\,(s^5+s^3) + 3167\,s^4\right)}{54\,u^3\,s\,(s-1)}$$

where

$$u^2 = s\,(8\,s^2 - 11\,s + 8).$$

(Equivalent to Dubrovin–Mazzocco's 10 page elliptic solution.)

24 branch Valentiner solution

(Icosahedral Solution 46)

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/3, 1/3, 1/3, 1/2)$$

$$y = \frac{1}{2} - \frac{\mathrm{P}}{2\,(3\,s^2 - 2\,s + 2)\,\mathrm{R}\,u}, \quad t = \frac{1}{2} + \frac{(s^2 + 4\,s - 2)\,\mathrm{Q}}{2\,(s+2)\,(3\,s^2 - 2\,s + 2)^2\,u^3}$$

where

$$\mathrm{P} = 16\,s^{11} + 72\,s^{10} + 50\,s^9 - 242\,s^8 - 3143\,s^7 + 6562\,s^6 - 8312\,s^5 + 9760\,s^4 - 9836\,s^3 + 6216\,s^2 - 2288\,s + 416,$$
$$\mathrm{Q} = 8\,s^{10} + 16\,s^9 + 24\,s^8 - 84\,s^7 + 429\,s^6 - 312\,s^5 + 258\,s^4 - 288\,s^3 + 288\,s^2 - 128\,s + 32,$$
$$\mathrm{R} = 26\,s^6 + 18\,s^5 - 75\,s^4 + 50\,s^3 + 270\,s^2 - 312\,s + 104,$$

and where (u, s) lies on the elliptic curve

$$u^2 = (8\,s^2 - 7\,s + 2)\,(s + 2).$$

Icosahedral solutions with $\geqslant 5$ branches

	Degree	Genus	Walls	A_5 Type	Alcove Point	No.	Group (size)
20	5	0	1	$b^2 c d$	44, 12, 12, 4	480	S_5
21	5	0	2	$c^2 d^2$	36, 12, 0, 0	240	S_5
22	6	0	1	$b c^2 d$	34, 10, 2, 2	576	S_6
23	6	0	1	$b c d^2$	46, 14, 10, 2	576	S_6
24	8	0	1	$a c^2 d$	39, 15, 3, 3	768	A_8
25	8	0	1	$a c d^2$	45, 9, 9, 3	768	A_8
26	9	1	2	$b c^3$	28, 4, 4, 4	288	A_9
27	9	1	2	$b d^3$	52, 8, 8, 4	288	A_9
28	10	0	2	$a^2 c d$	48, 12, 6, 6	480	$2^7\,3\,5$
29	10	0	2	$b^3 c$	46, 14, 14, 6	320	A_{10}
30	10	0	2	$b^3 d$	42, 2, 2, 2	320	A_{10}
31	10	0	3	c^4	24, 0, 0, 0	80	A_{10}
32	10	0	3	d^4	48, 0, 0, 0	80	A_{10}
33	12	0	0	$a b c d$	43, 11, 7, 1	2304	A_{12}
34	12	1	1	$a b c^2$	37, 13, 5, 5	1152	A_{12}
35	12	1	1	$a b d^2$	49, 5, 5, 1	1152	A_{12}
36	12	1	1	$b^2 c d$	38, 6, 6, 2	1152	$2^9\,3^2\,5$
37	15	1	2	$b^3 c$	36, 4, 4, 4	480	A_{15}
38	15	1	2	$b^3 d$	48, 8, 8, 8	480	A_{15}
39	15	1	2	$b^2 c^2$	32, 8, 0, 0	720	S_{15}
40	15	1	2	$b^2 d^2$	44, 4, 0, 0	720	S_{15}
41	18	1	3	b^4	40, 0, 0, 0	144	$2^{14}\,3^4\,5\,7$
42	20	1	1	$a b^2 c$	41, 9, 9, 1	1920	A_{20}
43	20	1	1	$a b^2 d$	47, 7, 3, 3	1920	A_{20}
44	20	1	3	$a^2 c^2$	42, 18, 0, 0	480	$2^{17}\,3^4\,5^2\,7$
45	20	1	3	$a^2 d^2$	54, 6, 0, 0	480	$2^{17}\,3^4\,5^2\,7$
46	24	1	2	$a b^3$	45, 5, 5, 5	768	$2^{20}\,3^5\,5^2\,7\,11$
47	30	2	2	$a^2 b c$	46, 14, 4, 4	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$
48	30	2	2	$a^2 b d$	52, 8, 2, 2	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$
49	36	3	3	$a^2 b^2$	50, 10, 0, 0	864	$2^{23}\,3^4\,5\,7$
50	40	3	3	$a^3 c$	51, 9, 9, 9	320	$2^{25}\,3^4\,5^2\,7$
51	40	3	3	$a^3 d$	57, 3, 3, 3	320	$2^{25}\,3^4\,5^2\,7$
52	72	7	3	$a^3 b$	55, 5, 5, 5	576	$2^{32}\,3^4\,5\,7$

Quadratic / Landen / Folding transformations

Kitaev   Manin   Tsuda-Okamoto-Sakai

Kitaev's perspective:

If A a fuchsian system with poles at $0, t, 1, \infty$
& with (proj.) monodromy of order 2 at $0, \infty$

- pullback A along $z \mapsto z^2$
- get system B with 4 non-apparent sing.s at $\pm 1, \pm\sqrt{t}$
- remove apparent sings & renormalize

* IMDs of A $\Leftrightarrow$ IMDs of resulting system *

$\rightsquigarrow$ get transform relating certain P_{VI} solutions
(codim 2 in param. space)

- Much simpler explicit formulae for transform later
(conjugate by Okamoto transformations)
(Ramani, Grammaticos, Tamizhmani 2000)

Icosahedral solutions with $\geqslant 5$ branches

	Degree	Genus	Walls	A_5 Type	Alcove Point	No.	Group (size)
20	5	0	1	b^2cd	44, 12, 12, 4	480	S_5
21	5	0	2	c^2d^2	36, 12, 0, 0	240	S_5
22	6	0	1	bc^2d	34, 10, 2, 2	576	S_6
23	6	0	1	bcd^2	46, 14, 10, 2	576	S_6
24	8	0	1	ac^2d	39, 15, 3, 3	768	A_8
25	8	0	1	acd^2	45, 9, 9, 3	768	A_8
26	9	1	2	bc^3	28, 4, 4, 4	288	A_9
27	9	1	2	bd^3	52, 8, 8, 4	288	A_9
28	10	0	2	a^2cd	48, 12, 6, 6	480	$2^7\,3\,5$
29	10	0	2	b^3c	46, 14, 14, 6	320	A_{10}
30	10	0	2	b^3d	42, 2, 2, 2	320	A_{10}
31	10	0	3	c^4	24, 0, 0, 0	80	A_{10}
32	10	0	3	d^4	48, 0, 0, 0	80	A_{10}
33	12	0	0	$abcd$	43, 11, 7, 1	2304	A_{12}
34	12	1	1	abc^2	37, 13, 5, 5	1152	A_{12}
35	12	1	1	abd^2	49, 5, 5, 1	1152	A_{12}
36	12	1	1	b^2cd	38, 6, 6, 2	1152	$2^9\,3^2\,5$
37	15	1	2	b^3c	36, 4, 4, 4	480	A_{15}
38	15	1	2	b^3d	48, 8, 8, 8	480	A_{15}
39	15	1	2	b^2c^2	32, 8, 0, 0	720	S_{15}
40	15	1	2	b^2d^2	44, 4, 0, 0	720	S_{15}
41	18	1	3	b^4	40, 0, 0, 0	144	$2^{14}\,3^4\,5\,7$
42	20	1	1	ab^2c	41, 9, 9, 1	1920	A_{20}
43	20	1	1	ab^2d	47, 7, 3, 3	1920	A_{20}
44	20	1	3	a^2c^2	42, 18, 0, 0	480	$2^{17}\,3^4\,5^2\,7$
45	20	1	3	a^2d^2	54, 6, 0, 0	480	$2^{17}\,3^4\,5^2\,7$
46	24	1	2	ab^3	45, 5, 5, 5	768	$2^{20}\,3^5\,5^2\,7\,11$
47	30	2	2	a^2bc	46, 14, 4, 4	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$
48	30	2	2	a^2bd	52, 8, 2, 2	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$
49	36	3	3	a^2b^2	50, 10, 0, 0	864	$2^{23}\,3^4\,5\,7$
50	40	3	3	a^3c	51, 9, 9, 9	320	$2^{25}\,3^4\,5^2\,7$
51	40	3	3	a^3d	57, 3, 3, 3	320	$2^{25}\,3^4\,5^2\,7$
52	72	7	3	a^3b	55, 5, 5, 5	576	$2^{32}\,3^4\,5\,7$

Solution 52

72 branches, genus 7

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/12, 1/12, 1/12, 11/12)$$

$$y = \frac{1}{2} + \frac{9\,(j-1)\,(j^3+27j^2-57j+79)\,wv + 2\,(2j^2-2j+5)\,(j^2-7j+1)\,(2j^4+2j^3-3j^2-58j+107)\,(j^2-4j+13)^2}{6\,(j^2-1)\,(2j^2+j+17)\,(j^3-3j^2+3j-11)\,(2j-7)^2\,v}$$

$$t = \frac{1}{2} + \frac{(s+1)\left(32\,(s^8+1) - 320\,(s^7+s) + 1112\,(s^6+s^2) - 2420\,(s^5+s^3) + 3167\,s^4\right)}{54\,s\,(s-1)\,u^3}$$

on the curve in $\mathbb{P}^3$ with affine equations:

$$v^2 = -\,(j+1)\left(6+j^2-2\,j\right)\left(4\,j^2-13\,j+19\right),$$
$$w^2 = (j-1)\,(2\,j-7)\,(j+1)\left(2\,j^2+j+17\right)\left(4\,j^2-13\,j+19\right)$$

where

$$s = \frac{j^2-1}{2j-7}, \qquad u = \frac{w}{(2j-7)^2}.$$

In fact this genus 7 curve is birational to the plane octic cut out by:

$$9\,(p^6\,q^2+p^2\,q^6)+18\,p^4\,q^4+4\,(p^6+q^6)+26\,(p^4\,q^2+p^2\,q^4)+8\,(p^4+q^4)+57\,p^2\,q^2+20\,(p^2+q^2)+16$$

Open problems

- Prove there are no more algebraic solutions

- Explain why all such solutions are defined / $\mathbb{Q}$

- Extend Hitchin's twistor approach to the icosahedral case

  ~ Umemura – Mukai's Fano 3-fold